Quantum Field Theory

M.Sc. 4th Semester

MPHYEC-1: Advanced Quantum Mechanics

Unit III (Part 1)

Dr. Ashok Kumar Jha

Assistant Professor

Department of Physics, Patna University

Mob:7903067108, Email: ashok.jha1984@gmail.com

Lecture Outline

- First and Second Quantization
- What is the need of Quantum Field Theory?
- Lagrangian formulation of Field
- Assignment

First Quantization

- Dynamical variables become operators;
- $E, L, \ldots$ take on only discrete values.

Particle Mechanics

$$\{q, p\}_{PB} = 1$$

q and p are coordinate & its conjugate momentum

→ Canonical Quantization Procedure →

Quantum Mechanics

$$[q, p] = i$$

q and p are operators in Hilbert space

Particle aspect of the object → First Quantization → Wave aspect of the object

Second Quantization

- Wave functions become field operators.
- Properties described by counting numbers of 1-particle states being occupied.
- Processes described in terms of exchange of real or virtual particles.
- Systems with variable numbers of particles ~ Second quantization
  - High energy scattering and decay processes.
  - Relativistic systems.
  - Many body systems (not necessarily relativistic)

What is Field?

- Classical analogue: Weighted vibrating string with infinite number of beads along the string, we come to limit of a continuous string described by a displacement field $\varphi(x,t)$ which varies continuously with $\boldsymbol{x}$ and $\boldsymbol{t}$.
- Field associated with each kind of particle in nature satisfies an assumed wave equation and it has infinite number of degrees of freedom.

A wave field is specified by its displacement at all points of space and the dependence of these displacements on time.

Road to Quantization of a Classical Field

(1) Start with the field equation

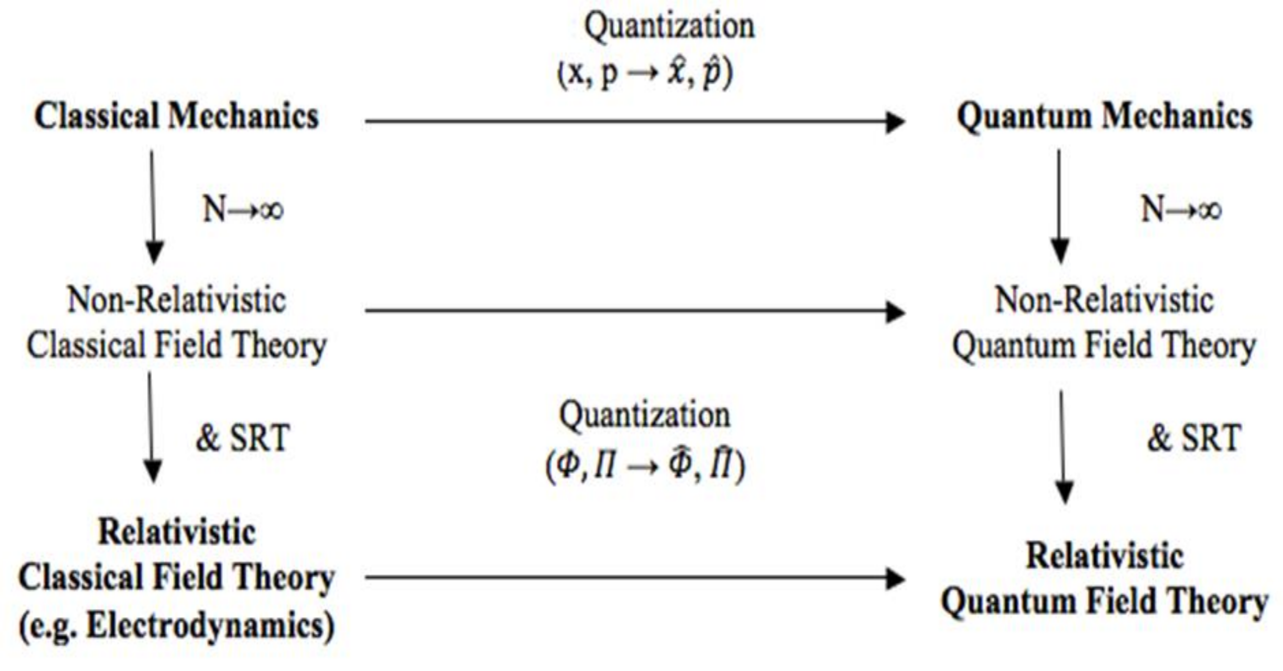
(2) Seek a Lagrangian via Hamilton's principle

(3) Find the canonical momentum from the Lagrangian

(4) Carry out quantization procedure

Why Quantum Field Theory?

- Quantum field theory is a theoretical framework that combines classical field theory, special relativity, and quantum mechanics but *not* general relativity's description of gravity.
- QFT is developed to study the behavior of particles moving with comparable to speed of light and the size is at atomic size.



Quantum Field Theory

Light is described in terms of

(1) electromagnetic field (classical description)

Quantum Field Theory establishes link between the two descriptions

(2) photon (quantum mechanical description)

Quantization of e m field → photon (quantum of the field)

- Structure to quantize the e m field is applicable to all elementary particles as quantum fields

Classical wave field — Second Quantization → Quantum of the field (particle)

Klein-Gordon Equation

Klein-Gordon equation for free particle is given by

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\varphi = 0$$ (using natural units $\hbar = c = 1$)

Energy eigen values are $E = \pm\sqrt{p^2 + m^2}$

Note: System will have no ground state and becomes unstable. Hence wave function interpretation of Klein-Gordon equation is not possible.

Now we want to interpret φ as a field, so Klein-Gordon eqn for this field is

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\varphi = 0$$

Consider an infinitesimal variation $\delta\varphi(x) = \varphi'(x) - \varphi(x)$

Restriction: $\delta\varphi(t_1) = \delta\varphi(t_2) = 0$ & system is localized in space

Real Scalar Field

From Klein-Gordon eqn $\int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} d^3x \left(\frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi + m^2 \varphi \right) \delta\varphi = 0$

Using the relation $\delta\left(\frac{\partial \varphi}{\partial x^\mu}\right) = \frac{\partial}{\partial x^\mu}(\varphi + \delta\varphi) - \frac{\partial}{\partial x^\mu}(\varphi) = \frac{\partial}{\partial x^\mu}(\delta\varphi)$

Integrating by parts $\int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} d^3x \; \delta\left[\frac{1}{2}\left\{ \left(\frac{\partial \varphi}{\partial t}\right)^2 - |\nabla\varphi|^2 - m^2\varphi^2 \right\} \right] = 0$

This gives $\delta \int_{t_1}^{t_2} d^4x \left[\frac{1}{2}\left(\frac{\partial \varphi}{\partial x^\mu} \frac{\partial \varphi}{\partial x_\mu} - m^2 \varphi^2 \right) \right] = 0$ with $x^\mu \equiv (t, \mathbf{x})$

or $\delta \int_{t_1}^{t_2} d^4x \mathscr{L}\left(\varphi, \frac{\partial \varphi}{\partial x^\mu} \right) = 0$ Hamilton's principle $\delta \int_{t_1}^{t_2} L dt = 0$

In analogy with Hamilton's principle

$$\mathscr{L}\left(\varphi, \frac{\partial \varphi}{\partial x^\mu} \right) = \frac{1}{2}\left(\frac{\partial \varphi}{\partial x^\mu} \frac{\partial \varphi}{\partial x_\mu} - m^2 \varphi^2 \right)$$

$\mathscr{L}$ is called Lagrangian density

Euler-Lagrange Equation

The momentum conjugate to the real scalar field is given by

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}$$

Hamilton's principle $\delta \int_{t_1}^{t_2} d^4x\, \mathcal{L} = 0$ with $\mathcal{L} = \mathcal{L}\left(\varphi, \frac{\partial \varphi}{\partial x^\mu}\right)$

This gives $\int_{t_1}^{t_2} d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x^\mu}\right)} \delta\left(\frac{\partial \varphi}{\partial x^\mu}\right) \right\} = 0$ using $\delta\left(\frac{\partial \varphi}{\partial x^\mu}\right) = \frac{\partial}{\partial x^\mu}(\delta\varphi)$

Integrating by parts $\int_{t_1}^{t_2} d^4x\ \delta\varphi \left(\frac{\partial \mathcal{L}}{\partial \varphi} + \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x^\mu}\right)} \right) = 0$

This leads to Euler-Lagrange equation

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x^\mu}\right)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

Assignment

1. Why we study quantum field theory?
2. What do you understand by First Quantization and Second Quantization?
3. Describe Lagrangian formulation of field equation.
4. Derive Euler-Lagrange Equation for field.

Thank You